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2005 J. Phys. A: Math. Gen. 38 L31

(http://iopscience.iop.org/0305-4470/38/2/L02)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) L31–L34 doi:10.1088/0305-4470/38/2/L02

LETTER TO THE EDITOR

An R-matrix for $D_4^{(3)}$

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Received 28 September 2004, in final form 19 October 2004
Published 15 December 2004
Online at stacks.iop.org/JPhysA/38/L31

Abstract
This letter gives the calculation of a trigonometric R-matrix for the twisted affine quantum group of type $D_4^{(3)}$.

PACS numbers: 02.20.Uw, 02.30.Ik

1. Introduction

This letter gives a new spectral decomposition of an R-matrix with spectral parameter. These R-matrices give the Hamiltonian in one-dimensional spin chain models, the Boltzmann weights in two dimensional solvable lattice models and the factorizable S-matrix in integrable quantum field theory.

The construction of these R-matrices is given in [Jim85] and [Jim86]. Let $U_q(\mathfrak{g})$ be the quantum group associated with an affine Dynkin diagram, D. This quantum group is a Hopf algebra defined by a finite presentation. The generators of this presentation are

$$\{e_i, h_i, f_i | i \in S\}$$

where S is the set of nodes of D.

Now choose a node $i \in S$ and let u be a parameter. Then, we have automorphisms, $\phi(u)$, of $U_q(\mathfrak{g})$ with

$$\phi(u)\colon e_i \mapsto u^2 e_i \qquad \phi(u)\colon f_i \mapsto u^{-2} f_i$$

and all other generators are unchanged. Now let V be a simple finite-dimensional $U_q(\mathfrak{g})$-module and denote the associated representation by $\rho_V\colon U_q(\mathfrak{g}) \to \mathrm{End}(V)$. Denote by $V(u)$ the $U_q(\mathfrak{g})$-module associated with the representation $\rho_V \cdot \phi(u)$. Let V and W be simple finite-dimensional $U_q(\mathfrak{g})$-modules; then the R-matrix, $\check{R}_{VW}(u)$, is determined by the Jimbo relations

$$\check{R}_{VW}(u) \cdot (\rho_{V(u)} \otimes \rho_W) \cdot \Delta(a) = (\rho_W \otimes \rho_{V(u)}) \cdot \Delta(a) \cdot \check{R}_{VW}(u) \tag{1}$$

for all $a \in U_q(\mathfrak{g})$ where $\Delta\colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ is the coproduct.

Let $U_q(\mathfrak{h})$ be the quantum group associated with a Dynkin diagram given by removing the node i from D. Note that this is the quantum group for a semisimple Lie algebra. Then $U_q(\mathfrak{h})$

is a subalgebra of $U_q(\mathfrak{g})$ and this inclusion is a homomorphism of Hopf algebras. The $U_q(\mathfrak{h})$-module obtained by restriction of the $U_q(\mathfrak{g})$-module $V(u)$ is independent of u. Hence, the R-matrix $\check{R}_{VW}(u)$ is an element of $\mathrm{End}_{U_q(\mathfrak{h})}(V\otimes W)$. This gives the following characterization of the R-matrix. It is an element of $\mathrm{End}_{U_q(\mathfrak{h})}(V\otimes W)$ which satisfies the Jimbo equations (1) for $a = e_i, h_i, f_i$. This means that if we are given the matrices representing the generators of $U_q(\mathfrak{g})$ then the problem of finding the R-matrix is reduced to solving a finite set of linear equations.

The canonical decomposition of $V\otimes W$ is

$$V\otimes W \cong \oplus_U\, M^U_{VW}\otimes U$$

where the sum is over a finite set of highest weight modules. Here, M^U_{VW} is a vector space whose dimension is the multiplicity of U in $V\otimes W$. Then the endomorphism algebra $\mathrm{End}_{U_q(\mathfrak{h})}(V\otimes W)$ is naturally isomorphic to

$$\oplus_U\, \mathrm{End}\left(M^U_{VW}\right)$$

and if we choose a basis of M^U_{VW} then we can identify endomorphisms with matrices. The general problem of giving an explicit description of an R-matrix then has two parts. The first part is to choose a basis of M^U_{VW} for each U so that the endomorphism algebra can be identified with a direct sum of matrix algebras. Using this identification, the R-matrix is a direct sum of matrices and so the second part of the problem is to give these matrices explicitly. In the opening paragraph, we loosely referred to this as the spectral decomposition of the R-matrix: this does determine the actual spectral decomposition of the R-matrix.

One additional requirement for this choice of basis is that the braid matrices as given in [GZ94] are diagonal.

The simplest situation occurs when all the vector spaces M^U_{VW} are one dimensional. In this case, the endomorphism algebra has a canonical basis of orthogonal idempotents. The R-matrix is a linear combination of these idempotents and the coefficients are given by the tensor product graph (or TPG). The tensor product graph is given in [DGZ94, GZ02, Mac91, ZGB91].

The tensor product graph method has been used to find R-matrices for the twisted affine Lie algebras $A^{(2)}_{2n}$ and $D^{(2)}_{n+1}$ in [DGZ96]. The tensor product graph method was further developed and used to find R-matrices for the twisted affine Lie algebras $A^{(2)}_{2n-1}$ and $E^{(2)}_6$ in [GMW96].

The only known spectral decompositions of R-matrices which cannot be obtained using this method are rational R-matrices for the representation $(\mathfrak{g}\oplus 1)$ of a simple Lie algebra $\mathfrak{g}$ in [CP91], and a universal R-matrices in [ZG94, BGZ95, ZG93, TK92].

There is a general method known as fusion of R-matrices (or the bootstrap procedure for factorised S-matrices) which will produce further spectral decompositions of R-matrices from a given one. Although there are applications of this method in the literature I am not aware of any R-matrices which have been computed by this method but which cannot be found using the tensor product graph method.

In this letter, we give an R-matrix with spectral parameter for the twisted affine Lie algebra $D^{(3)}_4$. Our method is to take the eight-dimensional representation given explicitly in [JM99] and then to solve the Jimbo relations directly using the computer algebra system Maple.

2. *R*-matrix

In this letter, we consider the case in which $U_q(\mathfrak{g})$ is the quantum group associated with the Dynkin diagram of type $D^{(3)}_4$. The node we remove is the one usually labelled by 0 so that the subalgebra $U_q(\mathfrak{h})$ is the quantum group associated with the Dynkin diagram G_2. Let V_8

be the eight-dimensional module given in [JM99]. Then in this section we give the R-matrix $R_{VV}(u)$.

Consider V_8 as a representation of $U_q(G_2)$. The R-matrix commutes with the action of $U_q(G_2)$ so we first describe a basis of the endomorphism algebra $\mathrm{End}_{U_q(G_2)}(V_8 \otimes V_8)$ and then we give the R-matrix with respect to this basis.

The representation V_8 decomposes as $V_7 \oplus 1$. The tensor product $V_8 \otimes V_8$ decomposes as $3V_7 \oplus 2.1 \oplus V_{14} \oplus V_{27}$. Let $\pi_1 \colon V_8 \to V_8$ be the projection onto the trivial representation and let π_7 be the projection onto V_7. Then, we get a decomposition of the identity of $\mathrm{End}_{U_q(G_2)}(V_8 \otimes V_8)$ into the four orthogonal idempotents $\pi_1 \otimes \pi_1$, $\pi_7 \otimes \pi_1$, $\pi_1 \otimes \pi_7$, $\pi_7 \otimes \pi_7$. The first three of these are primitive idempotents and the idempotent $\pi_7 \otimes \pi_7$ has a canonical decomposition into four orthogonal primitive idempotents since $V_7 \otimes V_7$ is a direct sum of four distinct simple modules. Denote these by $(\pi_7 \otimes \pi_7)_1$, $(\pi_7 \otimes \pi_7)_7$, $(\pi_7 \otimes \pi_7)_{14}$, $(\pi_7 \otimes \pi_7)_{27}$.

Let $\{E_{ij} \colon 1 \leqslant i, j \leqslant n\}$ be the basis of M_n consisting of elementary matrices. Then, we have a homomorphism $M_3 \to \mathrm{End}_{U_q(G_2)}(V_8 \otimes V_8)$ with $E_{11} \mapsto \pi_1 \otimes \pi_7$, $E_{22} \mapsto \pi_7 \otimes \pi_1$ and $E_{33} \mapsto (\pi_7 \otimes \pi_7)_7$.

Similarly, we have a homomorphism $M_2 \to \mathrm{End}_{U_q(G_2)}(V_8 \otimes V_8)$ with $E_{11} \mapsto \pi_1 \otimes \pi_1$ and $E_{22} \mapsto (\pi_7 \otimes \pi_7)_1$.

Here, we use the notation

$$[ax+b] = \frac{u^a q^b - u^{-a} q^{-b}}{q - q^{-1}}$$

which gives the usual q-integers when $a = 0$. If we take the limit $q \to 1$ then $[ax+b] \to ax+b$. This is the limit that gives the rational R-matrices.

The coefficient of $(\pi_7 \otimes \pi_7)_{14}$ is $[1-x]$ and the coefficient of $(\pi_7 \otimes \pi_7)_{27}$ is $[1+x]$.

The 2×2 matrix is

$$\frac{[x+2]}{[3x+6]} \frac{1}{[x+3]} \begin{pmatrix} A_{11} & \frac{[2x]}{[2]} \\ [2x][7]\frac{[3]^2[12]}{[4][6]} & A_{22} \end{pmatrix}$$

where

$$A_{11} = [x][3x+7] + [x+3][3x+2] + [2][3]\tfrac{[8]}{[4]}$$

$$A_{22} = [-x][-3x+7] + [-x+3][-3x+2] + [2][3]\tfrac{[8]}{[4]}.$$

The 3×3 matrix is given by

$$\frac{[x+2]}{[3x+6]} \begin{pmatrix} \frac{[3][2x+6]}{[2][x+3]} & [x]\left(\frac{[4x+6]}{[2x+3]} + \frac{[4]}{[2]^2}\right) & \frac{[x]}{[2]^2}([3]-3) \\ [x]\left(\frac{[4x+6]}{[2x+3]} + \frac{[4]}{[2]^2}\right) & \frac{[3][2x+6]}{[2][x+3]} & \frac{[x]}{[2]^2}([3]-3) \\ [x][3]^2\frac{[8]}{[4]} & [x][3]^2\frac{[8]}{[4]} & B_{33} \end{pmatrix}$$

where

$$B_{33} = -[3x-3] - [x-5] + \tfrac{1}{[2]}(2[x-4] - [2x+2]).$$

Here, we have $B_{21} = B_{12}$, $B_{13} = B_{23}$, $B_{31} = B_{32}$ and $B_{11} = B_{22}$. These are equivalent to the condition that this 3×3 matrix commutes with the permutation matrix which swaps the first two basis vectors. Hence, this matrix preserves the subspace spanned by $(1, -1, 0)$ and the subspace spanned by $(1, 1, 0)$ and $(0, 0, 1)$. This is the basis used in [CP91]. The eigenvalue of $(1, -1, 0)$ is $[1-x]$.

This R-matrix satisfies $R(1) = 1$ and the non-standard normalization $R(u)R(u^{-1}) = [1-x][1+x]$.

This R-matrix is not uniquely determined but can be modified in three distinct ways each of which introduces one parameter. First, the R-matrix can be multiplied by a scalar; next, we can conjugate the 2×2 matrix by a diagonal matrix with determinant 1; finally, we can conjugate the 3×3 matrix by a diagonal matrix with first two entries equal and determinant 1. These last two modifications correspond to scaling the basis; a canonical choice can be made by requiring that the R-matrix be symmetric but this involves introducing square roots.

If we multiply this R-matrix by

$$\frac{[3x+6]}{[x+2]}[x+3](q-q^{-1})^2$$

then we get matrices whose entries are Laurent polynomials in u with coefficients which are rational functions of q. The highest power of u with a non-zero coefficient is 4 and the lowest is -4. Taking the coefficients of u^4 we get a solution to the braid relation. The eigenvalues for the projections onto V_{14} and V_{27} are $-q^6$ and q^8, respectively, and the other two matrices are

$$\begin{pmatrix} q^6 & 0 \\ 0 & q^{-6} \end{pmatrix} \qquad \begin{pmatrix} 0 & q^6 & 0 \\ q^6 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Taking the coefficients of u^{-4} corresponds to substituting q^{-1} for q and gives the inverse matrices.

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